On partitions avoiding right crossings

Sherry H. F. Yan, Yuexiao Xu

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, P.R. China

huifangyan@hotmail.com

Abstract. Recently, Chen et al. derived the generating function for partitions avoiding right nestings and posed the problem of finding the generating function for partitions avoiding right crossings. In this paper, we derive the generating function for partitions avoiding right crossings via an intermediate structure of partial matchings avoiding 2-right crossings and right nestings. We show that there is a bijection between partial matchings avoiding 2-right crossing and right nestings and partitions avoiding right crossings.

KEY WORDS: Partial matchings, partitions, 2-right crossings, right nestings, right crossings.

AMS MATHEMATICAL SUBJECT CLASSIFICATIONS: 05A05, 05C30.

1 Introduction

This paper is concerned with the enumeration of set partitions that avoid right crossings. Recall that a partition π of $[n] = \{1, 2, \ldots, n\}$ can be represented as a diagram with vertices drawn on a horizontal line in increasing order. For a block B of π, we write the elements of B in increasing order. Suppose that $B = \{i_1, i_2, \ldots, i_k\}$. Then we draw an arc from i_1 to i_2, an arc from i_2 to i_3, and so on. Such a diagram is called the *linear representation* of π. If (i, j) is an arc in the diagram of π with $i < j$, we call vertex i an *opener*, and call vertex j a *transient* if there is an arc (j, k) for some integer k such that $i < j < k$; otherwise, we call vertex j a *closer*. A partial matching is a partition for which each block contains at most two elements. A partial matching without singletons is called a perfect matching.

A *nesting* of a partition is a pair of arcs (i_1, j_1) and (i_2, j_2) with $i_1 < i_2 < j_2 < j_1$. We call such a nesting a *left nesting* if $i_2 = i_1 + 1$. Similarly, we call it a *right nesting* if $j_1 = j_2 + 1$. A *crossing* of a partition is a pair of arcs (i_1, j_1) and (i_2, j_2) with $i_1 < i_2 < j_1 < j_2$ and we can define *left crossing* and *right crossing* analogously to how it was defined for nesting arcs. A *neighbor alignment* of a partition is a pair of arcs (i_1, j_1) and (i_2, j_2) such that $i_2 = j_1 + 1$.

Zagier [7] derived the generating function for perfect matchings avoiding left nestings and right nestings, which was introduced by Stoimenow [6]. Recently, Bousquet-Mélou et al. [1] constructed bijections between perfect matchings avoiding left nestings and right nestings and three other classes of combinatorial objects, that is, unlabeled $(2+2)$-free posets, permutations avoiding a certain pattern and ascent sequences. Dukes and Parviainen [4] presented a bijection between ascent sequences and upper triangular matrices whose non-negative entries

are such that all rows and columns contain at least one non-zero entry. Claesson and Linusson [3] established a bijection between perfect matchings avoiding left nestings and permutations and showed that perfect matchings avoiding left crossings are equinumerous with perfect matchings avoiding left nestings. They also introduced the notion of *k-left nesting* and conjectured that the number of perfect matchings of $[2n]$ without 2-left nestings is equal to the number of perfect matchings of $[2n]$ without left nestings and right nestings. This conjecture was confirmed by Levande [5]. Recall that two arcs (i_1, j_1) and (i_2, j_2) form a *k-left nesting* if $i_1 < i_2 < j_2 < j_1$ and $i_2 - i_1 \leq k$. Similarly, we call it a *k-right nesting* if $i_1 < i_2 < j_2 < j_1$ and $j_1 - j_2 \leq k$. Note that 1-left nesting is exactly a left nesting and 1-right nesting is exactly a right nesting. Similarly, we can define *k-left crossing* and *k-right crossing* analogously to how it was defined for nesting arcs. The left nestings, left crossings, right crossings, right crossings, neighbor alignments are called neighbor patterns.

Recently, Chen et al. [2] derived the generating functions for partial matchings avoiding neighbor alignments and for partial matchings avoiding neighbor alignments and left nestings. They obtained the generating function for partitions avoiding right nestings by presenting a bijection between partial matchings avoiding three neighbor patterns (left nestings, right nestings and neighbor alignments) and partitions avoiding right nestings. In general, the number of partitions of $[n]$ avoiding right crossings is not equal to the number of partitions of $[n]$ avoiding right nestings. In this paper, we derive the generating functions for partitions avoiding right crossings via an intermediate structure of partial matchings avoiding 2-right crossings and right nestings.

Denote by $\mathcal{M}(n,k)$ the set of partial matchings of $[n]$ with k arcs. The set of partial matchings in $\mathcal{M}(n,k)$ with no 2-right crossings and right nestings is denoted by $\mathcal{P}(n,k)$. Let $\mathcal{T}(n,k)$ be the set of partitions of $[n]$ with k arcs. Denote by $\mathcal{CT}(n,k)$ the set of partition in $\mathcal{T}(n,k)$ with no right crossings.

Denote by $P(n,k)$ and $CT(n,k)$ the cardinalities of the sets $\mathcal{P}(n,k)$ and $\mathcal{CT}(n,k)$, respectively.

We obtain the generating function formula for the numbers $P(n+k-1,k)$ by establishing a bijection between partial matchings avoiding 2-right crossings and right nestings and a certain class of integer sequence. Moreover, we show that there is a correspondence between $\mathcal{P}(n+k-1,k)$ and $\mathcal{CT}(n,k)$.

Theorem 1.1 *We have*

$$\sum_{n\geq 1}\sum_{k=0}^{n-1} P(n+k-1,k)x^n y^k = \sum_{n\geq 1}\frac{x^n(1+xy)^{\binom{n}{2}}}{\prod_{k=0}^{n-1}(1-(1+xy)^k xy)}. \tag{1.1}$$

Theorem 1.2 *We have*

$$\sum_{n\geq 1}\sum_{k=0}^{n-1} CT(n,k)x^n y^k = \sum_{n\geq 1}\frac{x^n(1+xy)^{\binom{n}{2}}}{\prod_{k=0}^{n-1}(1-(1+xy)^k xy)}. \tag{1.2}$$

2 Partial matchings avoiding 2-right crossings and right nestings

In this section, we construct a bijection between partial matchings avoiding 2-right crossings and right nestings and a certain class of integer sequences. As a consequence, we obtain the bivariate generating function for the number of partial matchings of $[n+k-1]$ with k arcs and containing no 2-right crossing and right nestings.

Let $x = x_1x_2\ldots x_n$ be an integer sequence. Denote by $max(x)$ the maximum element of x. For $1 < i \le n$, an element x_i of x is said to be a *left-to-right maximum* if $x_i > max(x_1x_2\ldots x_{i-1})$. For $n \ge 1$, let $\mathcal{S}(n,k)$ be the set of integer sequences $x = x_0x_1\ldots x_{n-1}$ with $n-k$ left-to-right maxima satisfying that

- $x_0 = 0$;
- for all $1 \le i \le n-1$, $0 \le x_i \le max(x_0x_1\ldots x_{i-1}) + 1$;
- for all $0 \le i \le n-1$, if $0 \le x_i < max(x_0x_1\ldots x_{i-1})$, then $x_i < x_{i-1}$.

Denote by $\mathcal{S}(n)$ the set of such integer sequences with n left-to-right maxima. Denote by $S(n,k)$ the cardinality of the set $\mathcal{S}(n,k)$. Let $f(x,y)$ be the generating function for the numbers $S(n,k)$. We derive the following generating function formula of $f(x,y)$.

Theorem 2.1

$$f(x,y) = \sum_{n\ge1}\sum_{k=0}^{n-1} S(n,k)x^ny^k = \sum_{n\ge1}\frac{x^n(1+xy)^{\binom{n}{2}}}{\prod_{k=0}^{n-1}(1-(1+xy)^kxy)}. \tag{2.1}$$

Proof. Let x be a sequence in $\mathcal{S}(n)$ with n left-to-right maxima. It can be uniquely decomposed as $0w_01w_12w_2\ldots(n-1)w_{n-1}$. For all $0 \le i \le n-1$, each w_i is a (possibly empty) integer sequence whose elements are nonnegative integers less than or equal to i. For all $1 \le i \le n-1$, suppose that there are i_l occurrences of i's in w_i, then w_i reads as $m_0im_1im_2\ldots im_{i_l}$, where for all $0 \le j \le i_l$, the sequence m_j is a (possibly empty) decreasing sequence whose elements are nonnegative integers less than i. Since w_0 is a (possibly empty) sequence of 0's. Hence, w_0 contributes $\frac{1}{1-xy}$ to the generating function $f(x,y)$, while for all $1 \le i \le n-1$, w_i contributes $\frac{(1+xy)^i}{1-(1+xy)^ixy}$ to the generating function $f(x,y)$. Furthermore, each left-to-right maximum contributes x to the generating function $f(x,y)$. Summing over all $n \ge 1$, we derive the generating function formula (2.1). This completes the proof. ∎

Now we proceed to construct a bijection between the set $\mathcal{P}(n+k-1,k)$ and the set $\mathcal{S}(n,k)$.

Theorem 2.2 *There exists a bijection between the set $\mathcal{P}(n+k-1,k)$ and the set $\mathcal{S}(n,k)$.*

Proof. Let M be a partial matching in $\mathcal{P}(n+k-1,k)$ of $[n+k-1]$ with k arcs and containing no 2-right crossings and right nestings. We wish to generate a sequence $\alpha(M)=x_0x_1x_2\dots x_{n-1}$ from M recursively. First, we remove the labels of the closers and relabel the vertices of M in the natural order. Denote by $\overline{M}$ the obtained diagram. For all $1\le i\le n-1$, let $\mathcal{O}(i)$ be the set of openers of arcs whose closers are left to vertex i of $\overline{M}$. Denote by $\mathcal{O}(n)$ the set of openers of all arcs of $\overline{M}$. Set $x_0=0$ and assume that we have obtained x_{i-1}. Now we proceed to generate x_i in the following manner.

Case 1. If there is no closer immediately after vertex i, then let $x_i=max(x_0x_1\dots x_{i-1})+1$.

Case 2. If there is an arc whose closer is immediately after vertex i and opener is labelled with j and j is the m-th minimum element of the set $[i]\setminus\mathcal{O}(i)$, then let $x_i=m-1$.

We claim that for all $0\le i\le n-1$, $max(x_0x_1x_2\dots x_i)=i-|\mathcal{O}(i+1)|$. We prove this claim by induction on i. Obviously, the claim holds for $i=0$. So let $i\ge 1$ and assume that the claim holds for $i-1$. In Case 1, we have $max(x_0x_1x_2\dots x_i)=max(x_0x_1\dots x_{i-1})+1=i-1-|\mathcal{O}(i)|+1$. Since there exist no closers immediately after vertex i in $\overline{M}$, we are led to $i-|\mathcal{O}(i+1)|=i-1-|\mathcal{O}(i)|+1=max(x_0x_1x_2\dots x_i)$. In Case 2, we have $0\le x_i\le i-|\mathcal{O}(i)|-1=max(x_0x_1\dots x_{i-1})$, which yields that $max(x_0x_1\dots x_i)=max(x_0x_1\dots x_{i-1})=i-1-|\mathcal{O}(i)|$. Since there is exactly one closer immediately after vertex i, we have $i-|\mathcal{O}(i+1)|=i-1-|\mathcal{O}(i)|=max(x_0x_1x_2\dots x_i)$. Therefore, the claim holds for all $0\le i\le n-1$. This yields that $0\le x_i\le max(x_0x_1x_2\dots x_{i-1})+1$ for all $1\le i\le n-1$ and $max(x_0x_1\dots x_{n-1})=n-1-|\mathcal{O}(n)|=n-1-k$, which implies that there are $n-k$ left-to-right maxima in $\alpha(M)$.

In order to prove that the obtained sequence $\alpha(M)\in\mathcal{S}(n,k)$, it remains to show that in the sequence $\alpha(M)$ if $0\le x_i<max(x_1x_2\dots x_{i-1})$, then $x_i<x_{i-1}$. According to the construction of the map α, there exists an arc whose closer is immediately after vertex i. Suppose that the opener of this arc is labelled with p in $\overline{M}$. Since $x_i<max(x_1x_2\dots x_{i-1})=i-1-|\mathcal{O}(i)|$, we are led to $p<i$. Obviously, when $x_{i-1}=max(x_1x_2\dots x_{i-1})$, we have $x_i<x_{i-1}$. Now suppose that $x_{i-1}<max(x_1x_2\dots x_{i-1})$. According to the construction of the map α, there exists an arc whose closer is immediately after vertex $i-1$. Suppose that the opener of this arc is labelled with q in $\overline{M}$. Since there are no 2-right crossings in M, we have $p<q$. It is easy to check that $\mathcal{O}(i)=\mathcal{O}(i-1)\cup\{q\}$. From the construction of the map α, we have $x_i<x_{i-1}$. Hence, the obtained sequence $\alpha(M)\in\mathcal{S}(n,k)$.

Conversely, given a sequence $x=x_0x_1x_2\dots x_{n-1}\in\mathcal{S}(n,k)$, we wish to construct a partial matching of $[n+k-1]$ with k arcs. First, we arrange $n-1$ vertices on a horizontal line and label them in the natural order. For $i=1,2,\dots,n-1$, at step i, we insert at most one closer after vertex i described as follows:

(i) if $x_i=max(x_0x_1x_2\dots x_{i-1})+1$, then we do nothing for vertex i;

(ii) if $0 \le x_i \le max(x_0x_1x_2\ldots x_{i-1})$, then we insert an arc whose opener is the (x_i+1)-th vacant vertex (from left to right) of the set $[i]$ and closer is immediately after vertex i, where a vacant vertex is a vertex which is not joined to any arc.

Finally, we relabel the vertices in the natural order to get a partial matching.

By induction on i $(0 \le i \le n-1)$, it is easy to see that after step i, the number of vacant vertices of the set $[i]$ is equal to $max(x_0x_1x_2\ldots x_i)$. So (ii) is valid and the construction of the inverse map is well defined. Since there are exactly $n-k$ left-to-right maxima in x, the obtained partial matching is a matching of $[n+k-1]$ with k arcs. Note that we add at most one closer immediately after each vertex i for all $1 \le i \le n-1$. Thus there is no two consecutive closers in the resulting partial matchings, which implies that there are no two right crossings and right nestings. Since when $0 \le x_i < max(x_1x_2\ldots x_{i-1})$, we have $x_i < x_{i-1}$, there are no 2-right crossings. Hence, the resulting partial matching is in the set $\mathcal{P}(n+k-1,k)$. This implies that the above map α is a bijection. ∎

For example, let $M = \{\{1,6\},\{2,3\},\{4,12\},\{5,10\},\{7,8\},\{9\},\{11\}\} \in \mathcal{P}(12,5)$. We obtain a diagram $\overline{M}$ from M by removing the labels of the closers and relabeling the vertex of M in the natural order, see Figure 1. We have $\mathcal{O}(1)=\emptyset$, $\mathcal{O}(2)=\emptyset$, $\mathcal{O}(3)=\{2\}$, $\mathcal{O}(4)=\{2\}, \mathcal{O}(5)=\{1,2\}, \mathcal{O}(6)=\{1,2,5\}$, $\mathcal{O}(7)=\{1,2,4,5\}$ and $\mathcal{O}(8)=\{1,2,3,4,5\}$. So the corresponding sequence is $\alpha(M)=01120210$.

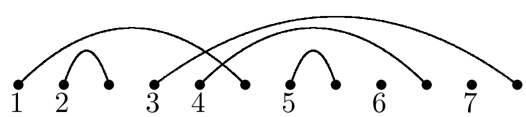

Figure 1: The diagram $\overline{M}$.

Conversely, given the integer sequence $x_0x_1x_2x_3x_4x_5x_6x_7 = 01120210 \in \mathcal{S}(8,5)$, the construction of the corresponding partial matching is illustrated in Figure 2.

Combining Theorems 2.1 and 2.2, we are led to the generating function formula (1.1).

3 Partitions with no right crossings

In this section, we present a bijection between the set $\mathcal{P}(n+k-1,k)$ and the set $\mathcal{CT}(n,k)$. As a result, we derive the generating functions for partitions avoiding right crossings. Before we present the bijection, we should recall the notion of 2-paths defined by Chen et al. [2]. Recall that a pair of two arcs (i,j) and (j,k) with $i<j<k$ in the diagram of a partition is said to be a 2-*path*.

Theorem 3.1 *There is a bijection between the set $\mathcal{P}(n+k-1,k)$ and the set $\mathcal{CT}(n,k)$. Moreover, this bijection transforms the number of neighbor alignments of a partial matching to the number of transients of a partition.*

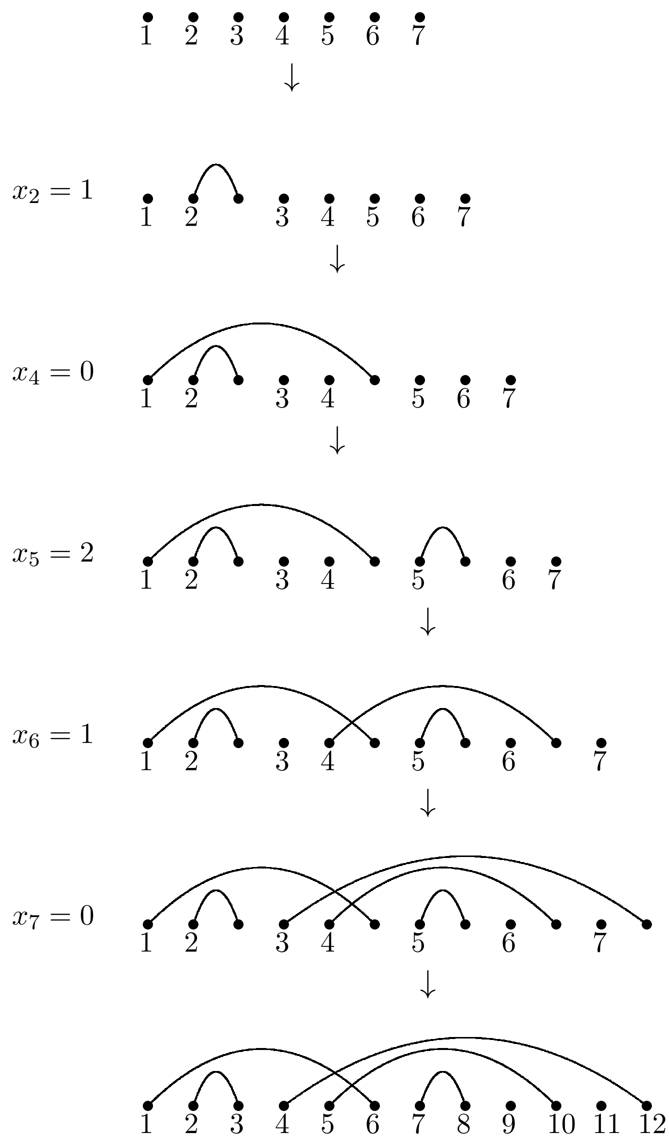

Figure 2: The bijection α.

Proof. Let M be a partial matching in $\mathcal{P}(n+k-1,k)$. we may reduce it to a partition by the following procedure.

- Change neighbor alignments to 2-paths from left to right until there are no more neighbor alignments, see Figure 3 for an illustration. More precisely, suppose that there is a neighbor alignment consisting of two arcs (i,j) and $(j+1,k)$. We change the arc $(j+1,k)$ to (j,k) and delete the vertex $j+1$.

- Delete the singleton immediately after each closer, except for the last closer.

- Relabel the vertices in the natural order.

Denote by P the obtained partition. We claim that there are no right crossings in the resulting partition P. Suppose that there are two crossing arcs (i,j) and $(k,j+1)$ with $i<k<j$ in the resulting partition. Since we have either deleted

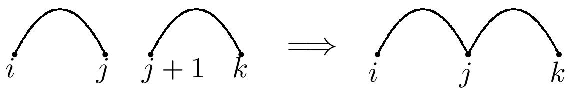

Figure 3: change a neighbor alignment to a 2-path.

an opener or a singleton immediately after each closer of M, except for the last closer, the two crossing arcs forms a 2-right crossing in the partial matching M, a contradiction. Hence, the claim is proved.

Conversely, given a partition $P' \in \mathcal{CT}(n,k)$, we wish to construct a partial matching $M' \in \mathcal{R}(n+k-1,k)$. First, we add a vertex after each closer, except for the last closer. Then, we change each pair of arcs (i,j) and (j,k) with $i<j<k$ into a neighbor alignment. Finally, we relabel the vertices in the natural order to get the partial matching M'.

From the construction of the inverse map, we know that we have either added a singleton or an opener after each closer of the partition P'. Hence, there are no right crossings and right nestings in the resulting partial matching. We claim that there are no 2-right crossings in the resulting partial matching M'. Suppose that there are two crossing arcs (i,j) and $(k,j+2)$ with $i<k<j$ in the resulting partial matching M'. Then $j+1$ is either a singleton or an opener. From the construction of the reverse map, the vertex $j+1$ is an added vertex. This implies that the two crossing arcs is a right crossing in the partition P', a contradiction. Hence, the claim is proved. This implies that there is a bijection between the set $\mathcal{P}(n+k-1,k)$ and the set $\mathcal{CT}(n,k)$. Obviously, this bijection transforms the number of neighbor alignments of a partial matching to the number of transients of a partition. This completes the proof. ∎

Combining Theorems 1.1 and 3.1, we derive the generating function Formula (1.2).

Figure 4 gives an example of a partial matching $M \in \mathcal{P}(12,5)$ and its corresponding partition $P \in \mathcal{CT}(8,5)$.

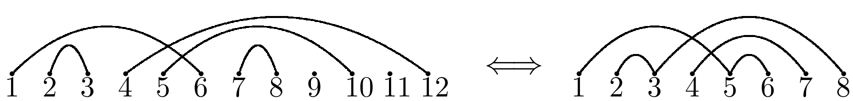

Figure 4: The partial matching M and its corresponding partition P.

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